

# Daisy cubes and distance cube polynomial

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Dedicated to the memory of our friend Michel Deza

## Abstract

Let  $X \subseteq \{0, 1\}^n$ . The daisy cube  $Q_n(X)$  is introduced as the subgraph of  $Q_n$  induced by the union of the intervals  $I(x, 0^n)$  over all  $x \in X$ . Daisy cubes are partial cubes that include Fibonacci cubes, Lucas cubes, and bipartite wheels. If  $u$  is a vertex of a graph  $G$ , then the distance cube polynomial  $D_{G,u}(x, y)$  is introduced as the bivariate polynomial that counts the number of induced subgraphs isomorphic to  $Q_k$  at a given distance from the vertex  $u$ . It is proved that if  $G$  is a daisy cube, then  $D_{G,0^n}(x, y) = C_G(x + y - 1)$ , where  $C_G(x)$  is the previously investigated cube polynomial of  $G$ . It is also proved that if  $G$  is a daisy cube, then  $D_{G,u}(x, -x) = 1$  holds for every vertex  $u$  in  $G$ .

**Keywords:** daisy cube; partial cube; cube polynomial; distance cube polynomial; Fibonacci cube; Lucas cube

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## 1 Introduction

In this paper we introduce a class of graphs, the members of which will be called daisy cubes. This new class contains several classes of graphs such as Fibonacci

cubes [18], Lucas cubes [26, 35], gear graphs (alias bipartite wheels) [16], and of course hypercubes themselves.

Our main motivation for the introduction of daisy cubes are the recent investigations of Saygı and Eğecioğlu [31, 32] in which they introduced the distance polynomial (alias  $q$ -cube polynomial) and studied it on Fibonacci cubes  $\Gamma_n$  and Lucas cubes  $\Lambda_n$ . This is a bivariate counting polynomial that keeps track of the number of subcubes that are at a given distance from the vertex  $0^n$  in  $\Gamma_n$  (resp.  $\Lambda_n$ ). Many of the results obtained in [31] and [32] present a refinement of the investigation [19] of the cube polynomial of Fibonacci cubes and Lucas cube. The latter polynomial is a counting polynomial of induced cubes in a graph; it was introduced in [3] and further studied in [4].

We proceed as follows. In the rest of this section we introduce some concepts and notation needed in this paper. In the next section we formally introduce daisy cubes, give several examples of them, and deduce some of their basic properties. In particular, daisy cubes admit isometric embeddings into hypercubes. In Section 3 we introduce the earlier investigated cube polynomial  $C_G(x)$  of a graph  $G$ , and the distance cube polynomial  $D_{G,u}(x, y)$ . In the main result of the section (Theorem 3.4) we prove a somehow surprising fact that the bivariate distance cube polynomial of an arbitrary daisy cube can be deduced from the univariate cube polynomial. More precisely, if  $G$  is a daisy cube, then  $D_{G,0^n}(x, y) = C_G(x+y-1)$ . Several consequences of this theorem are also developed. In particular, if  $G$  is a daisy cube, then the polynomials  $D_{G,0^n}$  and  $C_G$  are completely determined by the counting polynomial of the number of vertices at a given distance from the vertex  $0^n$ . In the final section we prove that  $D_{G,u}(x, -x) = 1$  holds for every vertex  $u$  of a daisy cube  $G$ .

Let  $B = \{0, 1\}$ . If  $u$  is a word of length  $n$  over  $B$ , that is,  $u = (u_1, \dots, u_n) \in B^n$ , then we will briefly write  $u$  as  $u_1 \dots u_n$ . The *weight* of  $u \in B^n$  is  $w(u) = \sum_{i=1}^n u_i$ , in other words,  $w(u)$  is the number of 1s in word  $u$ . We will use the power notation for the concatenation of bits, for instance  $0^n = 0 \dots 0 \in B^n$ .

The  $n$ -cube  $Q_n$  has the vertex set  $B^n$ , vertices  $u_1 \dots u_n$  and  $v_1 \dots v_n$  being adjacent if  $u_i \neq v_i$  for exactly one  $i \in [n]$ , where  $[n] = \{1, \dots, n\}$ . The set of all  $n$ -cubes is referred to as *hypercubes*. A *Fibonacci word* of length  $n$  is a word  $u = u_1 \dots u_n \in B^n$  such that  $u_i \cdot u_{i+1} = 0$  for  $i \in [n-1]$ . The *Fibonacci cube*  $\Gamma_n$ ,  $n \geq 1$ , is the subgraph of  $Q_n$  induced by the Fibonacci words of length  $n$ . A Fibonacci word  $u_1 \dots u_n$  is a *Lucas word* if in addition  $u_1 \cdot u_n = 0$  holds. The *Lucas cube*  $\Lambda_n$ ,  $n \geq 1$ , is the subgraph of  $Q_n$  induced by the Lucas words of length  $n$ . For convenience we also set  $\Gamma_0 = K_1 = \Lambda_0$ .

If  $u$  and  $v$  are vertices of a graph  $G$ , the *interval*  $I_G(u, v)$  between  $u$  and  $v$  (in  $G$ ) is the set of vertices lying on shortest  $u, v$ -path, that is,  $I_G(u, v) = \{w : d(u, v) = d(u, w) + d(w, v)\}$ . We will also write  $I(u, v)$  when  $G$  will be clear from the context. A subgraph  $H$  of a graph  $G$  is *isometric* if  $d_H(u, v) = d_G(u, v)$  holds for  $u, v \in V(H)$ . Isometric subgraphs of hypercubes are called *partial cubes*. For general properties of

these graphs we refer to the books [8, Chapter 19] and [27]. See also [1, 7, 22, 23] for a couple of recent developments on partial cubes and references therein for additional results. If  $H$  is a subgraph of a graph  $G$  and  $u \in V(G)$ , then the *distance*  $d(u, H)$  between  $u$  and  $H$  is  $\min_{v \in H}(d_G(u, v))$ . Finally, if  $G = (V(G), E(G))$  is a graph and  $X \subseteq V(G)$ , then  $\langle X \rangle$  denotes the subgraph of  $G$  induced by  $X$ .

## 2 Examples and basic properties of daisy cubes

Let  $\leq$  be a partial order on  $B^n$  defined with  $u_1 \dots u_n \leq v_1 \dots v_n$  if  $u_i \leq v_i$  holds for  $i \in [n]$ . For  $X \subseteq B^n$  we define the graph  $Q_n(X)$  as the subgraph of  $Q_n$  with

$$Q_n(X) = \langle \{u \in B^n : u \leq x \text{ for some } x \in X\} \rangle$$

and say that  $Q_n(X)$  is a *daisy cube* (generated by  $X$ ).

Vertex sets of daisy cubes are in extremal combinatorics known as *hereditary* or *downwards closed sets*, see [15, Section 10.2]. For instance, a result of Kleitman from [17] (cf. [15, Theorem 10.6]) reads as follows: If  $X, Y \subseteq B^n$  are hereditary sets, then  $|V(Q_n(X)) \cap V(Q_n(Y))| \geq |V(Q_n(X))| \cdot |V(Q_n(Y))|/2^n$ .

Before giving basic properties of daisy cubes let us list some of their important subclasses.

- If  $X = \{1^n\}$ , then  $Q_n(X) = Q_n$ .
- If  $X = \{u_1 \dots u_n : u_i \cdot u_{i+1} = 0, i \in [n-1]\}$ , then  $Q_n(X) = \Gamma_n$ .
- If  $X = \{u_1 \dots u_n : u_i \cdot u_{i+1} = 0, i \in [n-1], \text{ and } u_1 \cdot u_n = 0\}$ , then  $Q_n(X) = \Lambda_n$ .
- If  $X = \{110^{n-2}, 0110^{n-3}, \dots, 0^{n-2}11, 10^{n-1}1\}$ , then  $Q_n(X) = BW_n$  the bipartite wheel also known as a gear graph.
- If  $X = \{u : w(u) \leq n-1\}$ , then  $Q_n(X) = Q_n^-$  the vertex-deleted cube.

The above example which gives an equivalent description of Fibonacci cubes  $\Gamma_n$  can be rephrased by saying that  $X$  contains all words that do not contain the subword 11. This can be generalized by defining  $X_k$ ,  $k \geq 2$ , as the set of words that do not contain  $1^k$ . In this way more daisy cubes are obtained; in [21] these graphs were named generalized Fibonacci cubes. Today, the term “generalized Fibonacci cubes” is used for a much larger class of graphs as introduced in [13], see also [36] for an investigation of which generalized Fibonacci cubes are partial cubes.

Note that if  $x, y \in X$  and  $y \leq x$ , then  $Q_n(X) = Q_n(X \setminus \{y\})$ . More generally, if  $\hat{X}$  is the antichain consisting of the maximal elements of the poset  $(X, \leq)$ , then  $Q_n(\hat{X}) = Q_n(X)$ . Hence, for a given set  $X \subseteq B^n$  it is enough to consider the antichain  $\hat{X}$ ; we call the vertices of  $Q_n(X)$  from  $\hat{X}$  the *maximal vertices* of  $Q_n(X)$ .

For instance, let  $X = \{u \in B^n : w(u) \leq k\}$ . Then the maximal vertices of  $Q_n(X)$  are the vertices  $u$  with  $w(u) = k$ . In particular, the vertex-deleted  $n$ -cube  $Q_n^-$  can then be represented as

$$Q_n^- = Q_n(\{u : w(u) = n - 1\}).$$

If vertices  $u, v \in V(Q_n(X))$  differ in  $k$  coordinates, then it is straightforward to construct a  $u, v$ -path in  $Q_n(X)$  of length  $k$ . This immediately implies:

**Proposition 2.1** *If  $X \subseteq B^n$ , then  $Q_n(X)$  is a partial cube.*

The *isometric dimension*  $\text{idim}(G)$  of a partial cube  $G$  is the least integer  $n$  for which  $G$  embeds isometrically into  $Q_n$ . For an equivalent description of  $\text{idim}(G)$  recall that edges  $e = xy$  and  $f = uv$  of a graph  $G$  are in the relation  $\Theta$  if  $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$ . From [9, 37] we know that a connected graph  $G$  is a partial cube if and only if  $G$  is bipartite and  $\Theta$  is a transitive relation. Now,  $\text{idim}(G)$  is the number of  $\Theta$ -classes of  $G$ .

Another classical characterization of partial cubes is due to Chepoi: A graph  $G$  is a partial cube if and only if  $G$  can be obtained from the one vertex graph by a sequence of expansions [6]. To explain the result, let  $G_1$  and  $G_2$  be isometric subgraphs of  $G$  and let  $G_0 = G_1 \cap G_2 \neq \emptyset$ . Then the *expansion*  $H$  of  $G$  with respect to  $G_1$  and  $G_2$  is the graph obtained from the disjoint union of  $G_1$  and  $G_2$  by adding a matching between corresponding vertices in the two resulting copies of  $G_0$ . The inverse operation of an expansion in partial cubes is called *contraction*. In other words, a contraction of a partial cube is obtained by contracting the edges of a given  $\Theta$ -class.

After this preparation we can state two additional basic properties of daisy cubes.

**Proposition 2.2** *If  $G = Q_n(X)$  is a daisy cube, then  $\text{idim}(G) = \deg(0^n)$  and a contraction of  $G$  is a daisy cube.*

**Proof.** Edges incident to a fixed vertex of  $G$  lie in different  $\Theta$ -classes. Hence  $\text{idim}(G) \geq \deg(0^n)$ . On the other hand, if the end-vertices of an edge  $e \in E(G)$  differ in coordinate  $i$ , then  $e$  is in relation  $\Theta$  with the edge between  $0^n$  and  $0^{i-1}10^{n-i}$ . Thus  $\text{idim}(G) \leq \deg(0^n)$ .

Without loss of generality consider the contraction of  $G$  with respect to the  $\Theta$ -class containing the edge between  $0^n$  and  $10^{n-1}$ . Then every edge of  $G$  between vertices  $0w$  and  $1w$  (where  $w \in \{0, 1\}^{n-1}$ ) is contracted to the vertex  $w$ , while any other vertex  $u_1u_2 \dots u_n$  of  $G$  is in the contraction replaced by the vertex  $u_2 \dots u_n$ . It is now straightforward to conclude that the contraction is a daisy cube.  $\square$

The following observation will be important for our later studies.

**Lemma 2.3** *Let  $X \subseteq B^n$ . Then*

$$Q_n(X) = \left\langle \bigcup_{x \in \hat{X}} I_{Q_n}(x, 0^n) \right\rangle.$$

**Proof.** Let  $u \in V(Q_n(X))$ . We have already observed that  $Q_n(\hat{X}) = Q_n(X)$ , hence there exists a vertex  $x \in \hat{X}$  such that  $u \leq x$ . Then  $u \in I_{Q_n}(x, 0^n)$  and therefore  $V(Q_n(X)) = V(Q_n(\hat{X})) \subseteq \bigcup_{x \in \hat{X}} I_{Q_n}(x, 0^n)$ . Conversely, let  $u \in \bigcup_{x \in \hat{X}} I_{Q_n}(x, 0^n)$ . Then there exists a fixed vertex  $x \in \hat{X}$  such that  $u \in I_{Q_n}(x, 0^n)$ . But then  $u \leq x$  and consequently  $u \in V(Q_n(X))$  so that  $\bigcup_{x \in \hat{X}} I_{Q_n}(x, 0^n) \subseteq V(Q_n(X))$ .  $\square$

Lemma 2.3 is illustrated in Fig. 1. The figure also gives a clue why the name daisy cubes was selected.

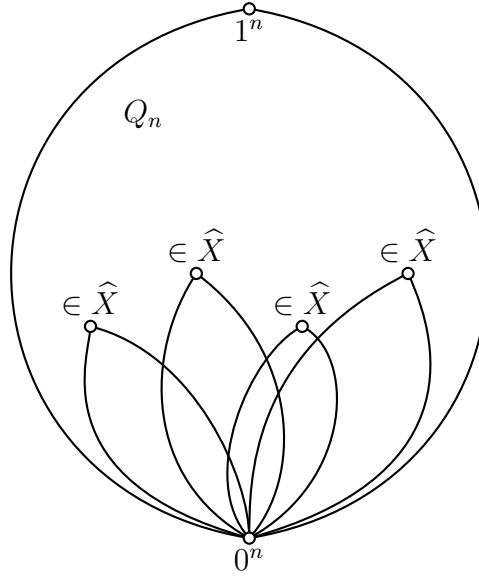


Figure 1: A daisy cube

### 3 Distance cube polynomial

Before introducing the cube polynomial and the distance cube polynomial we recall the Cartesian product operation and describe a structure of hypercubes in products.

The *Cartesian product*  $G \square H$  of graphs  $G$  and  $H$  has the vertex set  $V(G) \times V(H)$  and  $E(G \square H) = \{(g, h)(g', h') : gg' \in E(G) \text{ and } h = h', \text{ or, } g = g' \text{ and } hh' \in E(H)\}$ .

$E(H)\}$ . If  $(g, h) \in V(G \square H)$ , then the  $G$ -layer  $G^h$  through the vertex  $(g, h)$  is the subgraph of  $G \square H$  induced by the vertices  $\{(g', h) : g' \in V(G)\}$ . Similarly, the  $H$ -layer  ${}^gH$  through  $(g, h)$  is the subgraph of  $G \square H$  induced by the vertices  $\{(g, h') : h' \in V(H)\}$ . With  $p_G$  and  $p_H$  we denote projection maps from  $G \square H$  onto the factors  $G$  and  $H$ , respectively.

**Lemma 3.1** *If  $Q = Q_d$  is a subgraph of  $G \square H$ , then for some  $k \in [d - 1]$  we have  $p_G(Q) = Q_k$ ,  $p_H(Q) = Q_{d-k}$ , and hence  $Q = p_G(Q) \square p_H(Q)$ .*

**Proof.** Let  $(g, h) \in V(Q)$  and let  $(g_1, h), \dots, (g_k, h)$  and  $(g, h_1), \dots, (g, h_{d-k})$  be the neighbors of  $(g, h)$  in  $Q$ . Since the vertices  $(g, h), (g_1, h)$ , and  $(g_2, h)$  lie in a unique square of  $Q$ , the fourth vertex of this square must lie in  $G^h$ . Inductively applying this argument while having in mind the structure of  $Q$  we infer that the vertices  $(g, h), (g_1, h), \dots, (g_k, h)$  force an induced  $Q_k$  in the layer  $G^h$ , denote it with  $Q'$ . Similarly, the vertices  $(g, h), (g, h_1), \dots, (g, h_{d-k})$  force an induced  $Q_{d-k}$  in the layer  ${}^gH$ , denote it with  $Q''$ . Now, if  $e \in E(Q')$  and  $f \in E(Q'')$  are two incident edges, then there exists exactly one square in  $G \square H$  containing  $e$  and  $f$  and this square has no diagonals. (This fact is the so-called unique square property of the Cartesian product, see [12, Lemma 6.3].) Clearly, this square must lie in  $Q$ . Applying this argument for each pair of incident edges from  $Q'$  and  $Q''$  we conclude that  $Q = p_G(Q) \square p_H(Q)$ .  $\square$

For a graph  $G$  let  $c_k(G)$ ,  $k \geq 0$ , be the number of induced subgraphs of  $G$  isomorphic to  $Q_k$ , so that  $c_0(G) = |V(G)|$ ,  $c_1(G) = |E(G)|$ , and  $c_2(G)$  is the number of induced 4-cycles. The *cube polynomial*,  $C_G(x)$ , of  $G$ , is the corresponding counting polynomial, that is, the generating function

$$C_G(x) = \sum_{k \geq 0} c_k(G) x^k.$$

Since  $C_{K_2}(x) = 2 + x$  and  $Q_n$  is the Cartesian product of  $n$  copies of  $K_2$ , Lemma 3.1 yields:

$$C_{Q_n}(x) = (2 + x)^n. \quad (1)$$

In [31] a  $q$ -analogue of the cube polynomial of Fibonacci cubes  $\Gamma_n$  is considered with the remarkable property that this  $q$ -analogue counts the number of induced subgraphs isomorphic to  $Q_k$  at a given distance from the vertex  $0^n$ . (For related recent investigations on the number of disjoint hypercubes in Fibonacci cubes see [11, 24, 30].) We now introduce a generalization of this concept to arbitrary graphs as follows.

**Definition 3.2** *If  $u$  is a vertex of a graph  $G$ , then let  $c_{k,d}(G)$ ,  $k, d \geq 0$ , be the number of induced subgraphs of  $G$  isomorphic to  $Q_k$  at distance  $d$  from  $u$ . The*

distance cube polynomial of  $G$  with respect to  $u$  is

$$D_{G,u}(x, y) = \sum_{k,d \geq 0} c_{k,d}(G) x^k y^d.$$

For the later use we note that

$$D_{G,u}(x, 1) = C_G(x). \quad (2)$$

We also point out that if  $G$  is vertex-transitive, then  $D_{G,u}(x, y)$  is independent of  $u$ . Moreover, Lemma 3.1 implies:

**Proposition 3.3** *If  $G$  and  $H$  are graphs and  $(g, h) \in V(G \square H)$ , then*

$$D_{G \square H, (g, h)}(x, y) = D_{G, g}(x, y) D_{H, h}(x, y).$$

An immediate consequence of Proposition 3.3 is that if  $u \in V(Q_n)$ , then

$$D_{Q_n, u}(x, y) = D_{Q_n, 0^n}(x, y) = (1 + x + y)^n, \quad (3)$$

a result earlier obtained in [31].

We also point out that the class of daisy cubes is closed under the Cartesian product, a fact which further extends the richness of the class of daisy cubes.

Let  $H$  be an induced hypercube of  $Q_n$ . Then it is well-known that there exists a unique vertex of  $H$  with maximum weight, we will call it the *top vertex* of  $H$  and denote it with  $t(H)$ . Similarly,  $H$  contains a unique vertex with minimum weight to be called the *base vertex* of  $H$  and denoted  $b(H)$ . Furthermore  $H = \langle I(b(H), t(H)) \rangle$ .

We are now ready for the main result of this section.

**Theorem 3.4** *If  $G$  is a daisy cube, then  $D_{G, 0^n}(x, y) = C_G(x + y - 1)$ .*

**Proof.** Let  $G = Q_n(X)$  and  $\hat{X} = \{x_1, \dots, x_p\}$  be the maximal vertices of  $G$ . We thus have  $V(G) = \bigcup_{i \in [p]} I(0^n, x_i)$ .

An induced  $k$ -cube  $H$  of  $Q_n$  is an induced  $k$ -cube of  $G$  if and only if  $t(H) \in V(G)$ . Similarly an induced  $k$ -cube  $H$  of  $Q_n$  is an induced  $k$ -cube of  $\langle I(0^n, x) \rangle$  if and only if  $t(H) \in I(0^n, x)$ .

For any  $k$ -cube  $H$  of  $Q_n$  and any subset  $T$  of  $V(Q_n)$  let  $\mathbb{1}_H(T) = 1$  if  $t(H) \in T$ , and  $\mathbb{1}_H(T) = 0$  otherwise. Let  $\mathbb{H}_{k,d}$  be the set of induced  $k$ -cubes of  $Q_n$  that are at distance  $d$  from  $0^n$ , and let  $\mathbb{H}_k$  be the set of induced  $k$ -cubes of  $Q_n$ . Using this notation we have

$$D_{G, 0^n}(x, y) = \sum_k \sum_d \sum_{H \in \mathbb{H}_{k,d}} \mathbb{1}_H(V(G)) x^k y^d \quad (4)$$

and

$$C_G(z) = \sum_k \sum_{H \in \mathbb{H}_k} \mathbb{1}_H(V(G)) z^k. \quad (5)$$

By the inclusion-exclusion principle for the union of sets  $A_1, \dots, A_p$  we thus have

$$\mathbb{1}_H\left(\bigcup_{i \in [p]} A_i\right) = \sum_{J \subset [p], J \neq \emptyset} (-1)^{|J|-1} \mathbb{1}_H\left(\bigcap_{i \in J} A_i\right).$$

Therefore,

$$\mathbb{1}_H(V(G)) = \mathbb{1}_H\left(\bigcup_{i \in [p]} I(0^n, x_i)\right) = \sum_{J \subset [p], J \neq \emptyset} (-1)^{|J|-1} \mathbb{1}_H\left(\bigcap_{i \in J} I(0^n, x_i)\right).$$

Changing the order of summation in (4) and (5) we obtain

$$D_{G,0^n}(x, y) = \sum_{J \subset [p], J \neq \emptyset} (-1)^{|J|-1} D_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle, 0^n}(x, y)$$

and

$$C_G(z) = \sum_{J \subset [p], J \neq \emptyset} (-1)^{|J|-1} C_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle}(z).$$

Note that for arbitrary vertices  $u, v$  of  $Q_n$  we have  $I(0^n, u) \cap I(0^n, v) = I(0^n, u \wedge v)$ , where  $(u \wedge v)_i = 1$  if and only if  $u_i = 1$  and  $v_i = 1$ . The same property extends to the intersection of an arbitrary number of intervals. So  $\bigcap_{i \in J} I(0^n, x_i)$  is an interval that induces a hypercube with base vertex  $0^n$ . From (1) and (3) we see that the asserted result of the theorem holds if  $G$  is an induced hypercube with base vertex  $0^n$ . Therefore,

$$D_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle, 0^n}(x, y) = C_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle}(x + y - 1)$$

and we are done.  $\square$

Theorem 3.4 has the following immediate consequence.

**Corollary 3.5** *If  $G$  is a daisy cube, then  $D_{G,0^n}(x, y) = D_{G,0^n}(y, x)$ .*

This corollary in other words says that for any integers  $k, d$  the number of induced  $k$ -cubes at distance  $d$  from  $0^n$  in a daisy cube  $G$  is equal to the number of induced  $d$ -cubes at distance  $k$  from  $0^n$ .

To obtain another consequence of Theorem 3.4 we introduce the counting polynomial of the number of vertices at a given distance from a vertex  $u$  as follows.



**Definition 3.6** If  $u$  is a vertex of a graph  $G$ , then let  $w_d(G)$ ,  $d \geq 0$ , be the number of vertices of  $G$  at distance  $d$  from  $u$ . Then set

$$W_{G,u}(x) = \sum_{d \geq 0} w_d(G) x^d.$$

With this definition in hand we can state the following important consequence of Theorem 3.4.

**Corollary 3.7** If  $G$  is a daisy cube, then

$$D_{G,0^n}(x, y) = W_{G,0^n}(x + y) \quad \text{and} \quad C_G(x) = W_{G,0^n}(x + 1).$$

**Proof.** From Theorem 3.4 we get  $C_G(x) = D_{G,0^n}(0, x + 1)$ . Since by the definition of the polynomials  $D_{G,0^n}(0, x) = W_{G,0^n}(x)$  holds, and consequently  $D_{G,0^n}(0, x + 1) = W_{G,0^n}(x + 1)$ , we conclude that  $C_G(x) = W_{G,0^n}(x + 1)$ .

Using Theorem 3.4 again and the already proved second assertion of the corollary we get the first assertion:  $D_{G,0^n}(x, y) = C_G(x + y - 1) = W_{G,0^n}(x + y)$ .  $\square$

So if  $G$  is a daisy cube, then the polynomials  $D_{G,0^n}$  and  $C_G$  are completely determined by  $W_{G,0^n}$ .

Consider first the hypercube  $Q_n$ . Since the number of vertices of weight  $k$  in  $Q_n$  is  $\binom{n}{k}$ , we have  $W_{Q_n,0^n}(x) = (1 + x)^n$ . Hence from Corollary 3.7 we obtain again  $C_{Q_n}(x) = (2 + x)^n$  and  $D_{Q_n,0^n}(x, y) = (1 + x + y)^n$ .

For the Fibonacci cube  $\Gamma_n$  it is well-known that the number of vertices at distance  $k$  from  $0^n$  is  $\binom{n-k+1}{k}$ . Therefore  $W_{\Gamma_n,0^n}(x) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} x^k$  and we deduce that

$$C_{\Gamma_n}(x) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} (1+x)^k,$$

a result first proved in [19, Theorem 3.2] and that

$$\begin{aligned} D_{\Gamma_n,0^n}(x, y) &= \sum_{a=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-a+1}{a} (x+y)^a \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{d=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k-d+1}{k+d} \binom{k+d}{d} x^k y^d, \end{aligned}$$

a result obtained in [31, Proposition 3].

For the Lucas cube  $\Lambda_n$  we have  $W_{\Lambda_n, 0^n}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} [2\binom{n-k}{k} - \binom{n-k-1}{k}] x^k$ . Therefore,

$$C_{\Lambda_n}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[ 2\binom{n-k}{k} - \binom{n-k-1}{k} \right] (1+x)^k,$$

which is [19, Theorem 5.2]. We note in passing that in [19, Theorem 5.2] and [19, Corollary 5.3] there is a typo stating  $\binom{n-k+1}{k}$  instead of  $\binom{n-k-1}{k}$ . Moreover, Corollary 3.7 also gives that

$$D_{\Lambda_n, 0^n}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[ 2\binom{n-k}{k} - \binom{n-k-1}{k} \right] (x+y)^k.$$

## 4 A tree-like equality for daisy cubes

If  $G$  is a daisy cube, then the values of  $D_{G,u}(x, y)$  and  $W_{G,u}(x, y)$  depend on the choice of the vertex  $u$ . We demonstrate this on the vertex-deleted 3-cube  $Q_3^-$ .

$Q_3^-$  contains three orbits under the action of its automorphism group on the vertex set. Consider their representatives 000, 100, and 110, for which we have the following polynomials:

- $W_{Q_3^-, 000}(x) = 1 + 3x + 3x^2$ ,  
 $D_{Q_3^-, 000}(x, y) = 1 + 3y + 3y^2 + 3x + 6xy + 3x^2$ ;
- $W_{Q_3^-, 100}(x) = 1 + 3x + 2x^2 + x^3$ ,  
 $D_{Q_3^-, 100}(x, y) = 1 + 3y + 2y^2 + y^3 + x(3 + 4y + 2y^2) + x^2(2 + y)$ ;
- $W_{Q_3^-, 110}(x) = 1 + 2x + 3x^2 + x^3$ ,  
 $D_{Q_3^-, 110}(x, y) = 1 + 2y + 3y^2 + y^3 + x(2 + 4y + 3y^2) + x^2(1 + 2y)$ .

Note that  $D_{Q_3^-, u}(y, x) \neq D_{Q_3^-, u}(x, y)$  except for  $u = 0^n$ . In addition, there is no obvious relation between  $D_{Q_3^-, u}$  and  $W_{Q_3^-, u}$ . On the other hand,  $C_{Q_3^-}(x) = 7 + 9x + 3x^2$ , and we observe that  $D_{Q_3^-, u}(x, -x) = C_{Q_3^-}(-1) = 1$  holds for any vertex  $u$ .

Recall that a connected graph  $G$  is *median* if  $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$  holds for any triple of vertices  $u, v$ , and  $w$  of  $G$ . It is well-known that median graphs are partial cubes, cf. [20, Theorem 2]. Soltan and Chepoi [34] and independently Škrekovski [33] proved that if  $G$  is a median graph then  $C_G(-1) = 1$ . This equality in particular generalizes the fact that  $n(T) - m(T) = 1$  holds for a tree  $T$ . Hence if a daisy cube  $G$  is median (say a Fibonacci cube), then by Theorem 3.4 we have  $D_{G, 0^n}(x, -x) = 1$ . Our next result (to be proved in the rest of the section) asserts that this equality holds for every daisy cube and every vertex.

**Theorem 4.1** *If  $G$  is a daisy cube, then  $D_{G,u}(x, -x) = 1$  holds for every vertex  $u$  in  $G$ .*

The following consequence extends the class of partial cubes  $G$  for which  $C_G(-1) = 1$  holds.

**Corollary 4.2** *If  $G$  is a daisy cube, then  $C_G(-1) = 1$ .*

**Proof.** By Theorem 3.4 we have  $C_G(-1) = D_{G,0^n}(x, -x)$ . Theorem 4.1 completes the argument.  $\square$

In the rest the following concept from [10] (see also [5]) will be useful. A subgraph  $H$  of a graph  $G$  is called *gated* if for every  $u \in V(G)$ , there exists a vertex  $x \in V(H)$  such that for every  $v \in V(H)$  the vertex  $x$  lies on a shortest  $u, v$ -path. If such a vertex exists, it must be unique and is denoted  $\pi_H(u)$ . For us it is important that sub-hypercubes in partial cubes are gated, cf. [29, p. 2122]. (We note in passing that Berrachedi [2] characterized median graphs as the connected graphs  $G$  in which intervals induce gated subgraphs.)

Recall that if  $u, v \in V(Q_n)$ , then  $u \wedge v$  is the vertex with  $(u \wedge v)_i = 1$  if and only if  $u_i = 1$  and  $v_i = 1$ . The following fact is straightforward.

**Lemma 4.3** *If  $u$  and  $v$  are vertices of  $Q_n$  and  $G = \langle I(0^n, v) \rangle$ , then  $\pi_G(u) = u \wedge v$ .*

**Lemma 4.4** *Let  $c \in V(Q_n)$  and let  $u$  be a vertex from  $I(0^n, c)$ . Then for any vertex  $b$  of  $Q_n$  we have  $\pi_{\langle I(0^n, c) \cap I(0^n, b) \rangle}(u) = \pi_{\langle I(0^n, b) \rangle}(u)$ .*

**Proof.** Since  $u$  belongs to  $I(0^n, c)$  we have  $u \wedge c = u$  and  $I(0^n, c) \cap I(0^n, b) = I(0^n, c \wedge b)$ . Therefore, having in mind Lemma 4.3,

$$\begin{aligned} \pi_{\langle I(0^n, c) \cap I(0^n, b) \rangle}(u) &= \pi_{\langle I(0^n, c \wedge b) \rangle}(u) \\ &= u \wedge c \wedge b = u \wedge b \\ &= \pi_{\langle I(0^n, b) \rangle}(u). \end{aligned}$$

$\square$

Let  $u$  be vertex of  $Q_n$  and let  $G$  be a *fixed* subgraph of  $Q_n$ . Then we can naturally extend the definition of the distance cube polynomial  $D_{G,u}(x, y)$  also to the case when  $u \notin V(G)$ . Note that in order that  $D_{G,u}(x, y)$  is well-defined in such a case, together with  $u$  and  $G$  we also need the embedding of  $G$  into  $Q_n$ .

**Lemma 4.5** *If  $u, b \in V(Q_n)$  and  $G = \langle I(0^n, b) \rangle$ , then*

$$D_{G,u}(x, -x) = (-x)^{d(u, G)}.$$

**Proof.** The contribution of some induced  $k$ -cube  $H$  of  $G$  to the polynomial  $D_{G,u}(x, y)$  is  $x^k y^\delta$  where  $\delta = d(u, H)$ . Because intervals are gated in median graphs and hence in hypercubes, see [20, Theorem 6(vi)], we have

$$\delta = d(u, \pi_G(u)) + d(\pi_G(u), H).$$

Therefore  $D_{G,u}(x, y) = y^{d(u, G)} D_{G, \pi_G(u)}(x, y)$ . Since  $G$  is a hypercube, thus a vertex-transitive graph, and  $\pi_G(u)$  belongs to  $G$ , we have

$$D_{G, \pi_G(u)}(x, -x) = D_{G, 0^n}(x, -x) = 1.$$

We conclude that  $D_{G,u}(x, -x) = (-x)^{d(u, G)}$ .  $\square$

**Proof (of Theorem 4.1).** Assume that  $G = Q_n(\widehat{X})$  with  $\widehat{X} = \{x_i, i \in I\}$  thus  $V(G) = \bigcup_{i \in I} I(0^n, x_i)$ . By inclusion-exclusion formula,

$$D_{G,u}(x, -x) = \sum_{J \subset I, J \neq \emptyset} (-1)^{|J|-1} D_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle, u}(x, -x). \quad (6)$$

Since  $\bigcap_{i \in J} I(0^n, x_i)$  is some interval  $I(0^n, b_J)$ , Lemma 4.5 implies that

$$D_{\langle \bigcap_{i \in J} I(0^n, x_i) \rangle, u}(x, -x) = (-x)^{d(u, \langle \bigcap_{i \in J} I(0^n, x_i) \rangle)}.$$

Let  $i_0 \in I$  such that  $u$  belong to  $I(0^n, x_{i_0})$  and let  $I' = I \setminus \{i_0\}$ . For every subset  $J'$  of  $I'$  consider the pair  $\{J', J' \cup \{i_0\}\}$ . We obtain the following partition of the power set  $\mathcal{P}(I)$

$$\mathcal{P}(I) = \bigcup_{J' \subset I'} \{J' \cup (J' \cup \{i_0\})\}.$$

If  $J'$  is not empty then by Lemmas 4.3 and 4.4 we have

$$d(u, \langle \bigcap_{i \in J' \cup \{i_0\}} I(0^n, x_i) \rangle) = d(u, \langle \bigcap_{i \in J'} I(0^n, x_i) \rangle),$$

and since  $|J' \cup \{i_0\}| = |J'| + 1$ , the sum of contribution in equation (6) of  $J'$  and  $J' \cup \{i_0\}$  to  $D_{G,u}(x, -x)$  is null. Therefore the only term remaining corresponds to the pair  $\{\emptyset, \{i_0\}\}$ . Thus  $D_{G,u}(x, -x) = (-x)^{d(u, \langle I(0^n, x_{i_0}) \rangle)} = 1$ .  $\square$

## 5 Concluding remarks

In this paper we have introduced daisy cubes and showed that they possess some appealing properties. Further investigation of the structure of daisy cubes would be in place and seems interesting. We pose here some related open problems.

As already mentioned, Chepoi [6] characterized partial cubes as the graphs obtainable from  $K_1$  by a sequence of expansions. Such a characterization was earlier done for the case of median graphs in [25] and later for different additional subclasses of partial cubes, cf. [7, 14, 28, 29].

**Problem 5.1** *Do daisy cubes admit a characterization in terms of an expansion procedure?*

A positive answer to the next question would yield another characterization of daisy cubes.

**Problem 5.2** *Does the converse of Theorem 3.4 hold?*

Similarly, we also ask:

**Problem 5.3** *Does the second equality of Corollary 3.7 implies that  $G$  is a daisy cubes, that is, are daisy cubes characterized by this equality?*

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